

Non-Commutative Worlds and Classical Constraints

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1 Introduction

Aspects of gauge theory, Hamiltonian mechanics, relativity and quantum mechanics arise naturally in the mathematics of a non-commutative framework for calculus and differential geometry. In this paper, we first give a review of our previous results about discrete physics and non-commutative worlds. The simplest discrete system corresponds directly to the square root of minus one, seen as an oscillation between one and minus one. This way thinking about i as an *iterant* is explained below. By starting with a discrete time series of positions, one has immediately a non-commutativity of observations since the measurement of velocity involves the tick of the clock and the measurement of position does not demand the tick of the clock. Commutators that arise from discrete observation suggest a non-commutative calculus, and this calculus leads to a generalization of standard advanced calculus in terms of a non-commutative world. In a non-commutative world, all derivatives are represented by commutators. We then give our version of Feynman-Dyson derivation of the formalism of electromagnetic gauge theory. The rest of the paper investigates algebraic constraints that bind the commutative and non-commutative worlds.

Section 2 is a self-contained miniature version of the whole story in this paper, starting with the square root of minus one seen as a discrete oscillation, a clock. We proceed from there and analyze the position of the square root of minus one in relation to discrete systems and quantum mechanics. We end this section by fitting together these observations into the structure of the Heisenberg commutator

$$[p, q] = i\hbar.$$

Section 3 is a review of the context of non-commutative worlds with a preview of part of the Feynman-Dyson derivation. This section generalizes the concepts in Section 2 and places them in the wider context of non-commutative worlds. The key to this generalization is our method of embedding discrete calculus into non-commutative calculus. Section 4 is a discussion of iterants

and matrix algebra. We show how matrix algebra in any dimension can be regarded as describing the pattern of acts of observation (time shifting operators corresponding to permutations) on periodic time series. Section 5 gives a complete treatment of our generalization of the Feynman-Dyson derivation of Maxwell's equations in a non-commutative framework. This section is the first foray into the consequences of constraints. This version of the Feynman-Dyson derivation depends entirely on the postulation of a full time derivative in the non-commutative world that matches the corresponding formula in ordinary commutative advanced calculus. Section 6 discusses constraints on non-commutative worlds that are imposed by asking for correspondences between forms of classical differentiation and the derivatives represented by commutators in a correspondent non-commutative world. This discussion of constraints parallels work of Tony Deakin [3, 4] and will be continued in joint work of the author and Deakin. At the level of the second constraint we encounter issues related to general relativity. Section 7 continues the constraints discussion in Section 5, showing how to generalize to higher-order constraints and obtains a commutator formula for the third order constraint. The first appendix, Section 8, is a very condensed review of the relationship of the Bianchi identity in differential geometry and the Einstein equations for general relativity. We then observe that every derivation in a non-commutative world comes equipped with its own Bianchi identity. This observation suggests one way to investigate general relativity in the non-commutative context. Section 9 is a philosophical appendix.

2 Quantum Mechanics - The Square Root of Minus One is a Clock

The purpose of this section is to place i , the square root of minus one, and its algebra in a context of discrete recursive systems. We begin by starting with a simple periodic process that is associated directly with the classical attempt to solve for i as a solution to a quadratic equation. We take the point of view that solving $x^2 = ax + b$ is the same (when $x \neq 0$) as solving

$$x = a + b/x,$$

and hence is a matter of finding a fixed point. In the case of i we have

$$x^2 = -1$$

and so desire a fixed point

$$x = -1/x.$$

There are no real numbers that are fixed points for this operator and so we consider the oscillatory process generated by

$$R(x) = -1/x.$$

The fixed point would satisfy

$$i = -1/i$$

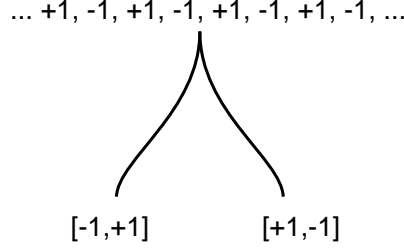


Figure 1: **A Basic Oscillation**

and multiplying, we get that

$$ii = -1.$$

On the other hand the iteration of R yields

$$1, R(1) = -1, R(R(1)) = +1, R(R(R(1))) = -1, +1, -1, +1, -1, \dots$$

The square root of minus one is a perfect example of an eigenform that occurs in a new and wider domain than the original context in which its recursive process arose. The process has no fixed point in the original domain.

Looking at the oscillation between $+1$ and -1 , we see that there are naturally two phase-shifted viewpoints. We denote these two views of the oscillation by $[+1, -1]$ and $[-1, +1]$. These viewpoints correspond to whether one regards the oscillation at time zero as starting with $+1$ or with -1 . See Figure 1.

We shall let $I\{+1, -1\}$ stand for an undisclosed alternation or ambiguity between $+1$ and -1 and call $I\{+1, -1\}$ an iterant. There are two iterant views: $[+1, -1]$ and $[-1, +1]$.

Given an iterant $[a, b]$, we can think of $[b, a]$ as the same process with a shift of one time step. These two iterant views, seen as points of view of an alternating process, will become the square roots of negative unity, i and $-i$.

We introduce a temporal shift operator η such that

$$[a, b]\eta = \eta[b, a]$$

and

$$\eta\eta = 1$$

for any iterant $[a, b]$, so that concatenated observations can include a time step of one-half period of the process

$$\dots abababab \dots$$

We combine iterant views term-by-term as in

$$[a, b][c, d] = [ac, bd].$$

We now define i by the equation

$$i = [1, -1]\eta.$$

This makes i both a value and an operator that takes into account a step in time.

We calculate

$$ii = [1, -1]\eta[1, -1]\eta = [1, -1][-1, 1]\eta\eta = [-1, -1] = -1.$$

Thus we have constructed the square root of minus one by using an iterant viewpoint. In this view i represents a discrete oscillating temporal process and it is an eigenform for $R(x) = -1/x$, participating in the algebraic structure of the complex numbers. In fact the corresponding algebra structure of linear combinations $[a, b] + [c, d]\eta$ is isomorphic with 2×2 matrix algebra and iterants can be used to construct $n \times n$ matrix algebra. We treat this generalization in Section 4 of this paper.

The Temporal Nexus. *We take as a matter of principle that the usual real variable t for time is better represented as it so that time is seen to be a process, an observation and a magnitude all at once.* This principle of “imaginary time” is justified by the eigenform approach to the structure of time and the structure of the square root of minus one.

As an example of the use of the Temporal Nexus, consider the expression $x^2 + y^2 + z^2 + t^2$, the square of the Euclidean distance of a point (x, y, z, t) from the origin in Euclidean four-dimensional space. Now replace t by it , and find

$$x^2 + y^2 + z^2 + (it)^2 = x^2 + y^2 + z^2 - t^2,$$

the squared distance in hyperbolic metric for special relativity. By replacing t by its process operator value it we make the transition to the physical mathematics of special relativity.

2.1 Quantum Physics, Eigenvalue and Eigenform

In quantum modeling, the state of a physical system is represented by a vector in a Hilbert space. As time goes on the vector undergoes a unitary evolution in the Hilbert space. Observable quantities correspond to Hermitian operators H and vectors v that have the property that the application of H to v results in a new vector that is a multiple of v by a real factor λ . Thus

$$Hv = \lambda v.$$

One says that v is an eigenvector for the operator H , and that λ is the eigenvalue.

2.2 The Wave Function in Quantum Mechanics and The Square Root of Minus One

One can regard a wave function such as $\psi(x, t) = \exp(i(kx - wt))$ as containing a micro-oscillatory system with the special synchronizations of the iterant view $i = [+1, -1]\eta$. It is these

synchronizations that make the big eigenform of the exponential work correctly with respect to differentiation, allowing it to create the appearance of rotational behaviour, wave behaviour and the semblance of the continuum. In other words, we are suggesting that once can take a temporal view of the well-known equation of Euler:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

by regarding the i in this equation as an iterant, as discrete oscillation between -1 and $+1$. One can blend the classical geometrical view of the complex numbers with the iterant view by thinking of a point that orbits the origin of the complex plane, intersecting the real axis periodically and producing, in the real axis, a periodic oscillation in relation to its orbital movement in the two dimensional space. The special synchronization is the algebra of the time shift embodied in

$$\eta\eta = 1$$

and

$$[a, b]\eta = \eta[b, a]$$

that makes the algebra of $i = [1, -1]\eta$ imply that $i^2 = -1$. This interpretation does not change the formalism of these complex-valued functions, but it does change one's point of view and we now show how the properties of i as a discrete dynamical system are found in any such system.

2.3 Time Series and Discrete Physics

We have just reformulated the complex numbers and expanded the context of matrix algebra to an interpretation of i as an oscillatory process and matrix elements as combined spatial and temporal oscillatory processes (in the sense that $[a, b]$ is not affected in its order by a time step, while $[a, b]\eta$ includes the time dynamic in its interactive capability, and 2×2 matrix algebra is the algebra of iterant views $[a, b] + [c, d]\eta$).

We now consider elementary discrete physics in one dimension. Consider a time series of positions

$$x(t) : t = 0, \Delta t, 2\Delta t, 3\Delta t, \dots$$

We can define the velocity $v(t)$ by the formula

$$v(t) = (x(t + \Delta t) - x(t))/\Delta t = Dx(t)$$

where D denotes this discrete derivative. In order to obtain $v(t)$ we need at least one tick Δt of the discrete clock. Just as in the iterant algebra, we need a time-shift operator to handle the fact that once we have observed $v(t)$, the time has moved up by one tick.

We adjust the discrete derivative. We shall add an operator J that in this context accomplishes the time shift:

$$x(t)J = Jx(t + \Delta t).$$

We then redefine the derivative to include this shift:

$$Dx(t) = J(x(t + \Delta t) - x(t))/\Delta t.$$

This readjustment of the derivative rewrites it so that the temporal properties of successive observations are handled automatically.

Discrete observations do not commute. Let A and B denote quantities that we wish to observe in the discrete system. Let AB denote the result of first observing B and then observing A . The result of this definition is that a successive observation of the form $x(Dx)$ is distinct from an observation of the form $(Dx)x$. In the first case, we first observe the velocity at time t , and then x is measured at $t + \Delta t$. In the second case, we measure x at t and then measure the velocity.

We measure the difference between these two results by taking a commutator

$$[A, B] = AB - BA$$

and we get the following computations where we write $\Delta x = x(t + \Delta t) - x(t)$.

$$x(Dx) = x(t)J(x(t + \Delta t) - x(t)) = Jx(t + \Delta t)(x(t + \Delta t) - x(t)).$$

$$(Dx)x = J(x(t + \Delta t) - x(t))x(t).$$

$$[x, Dx] = x(Dx) - (Dx)x = (J/\Delta t)(x(t + \Delta t) - x(t))^2 = J(\Delta x)^2/\Delta t$$

This final result is worth recording:

$$[x, Dx] = J(\Delta x)^2/\Delta t.$$

From this result we see that the commutator of x and Dx will be constant if $(\Delta x)^2/\Delta t = K$ is a constant. For a given time-step, this means that

$$(\Delta x)^2 = K\Delta t$$

so that

$$\Delta x = \pm\sqrt{(K\Delta t)}$$

This is a Brownian process with diffusion constant equal to K .

Thus we arrive at the result that any discrete process viewed in this framework of discrete observation has the basic commutator

$$[x, Dx] = J(\Delta x)^2/\Delta t,$$

generalizing a Brownian process and containing the factor $(\Delta x)^2/\Delta t$ that corresponds to the classical diffusion constant. It is worth noting that the adjustment that we have made to the discrete derivative makes it into a commutator as follows:

$$Dx(t) = J(x(t + \Delta t) - x(t))/\Delta t = (x(t)J - Jx(t))\Delta t = [x(t), J]/\Delta t.$$

By replacing discrete derivatives by commutators we can express discrete physics in many variables in a context of non-commutative algebra. We enter this generalization in the next section of the paper.

We now use the temporal nexus (the square root of minus one as a clock) and rewrite these commutators to match quantum mechanics.

2.4 Simplicity and the Heisenberg Commutator

Finally, we arrive at the simplest place. Time and the square root of minus one are inseparable in the temporal nexus. The square root of minus one is a symbol and algebraic operator for the simplest oscillatory process. As a symbolic form, i is an eigenform satisfying the equation

$$i = -1/i.$$

One does not have an increment of time all alone as in classical t . One has it , a combination of an interval and the elemental dynamic that is time. With this understanding, we can return to the commutator for a discrete process and use it for the temporal increment.

We found that discrete observation led to the commutator equation

$$[x, Dx] = J(\Delta x)^2/\Delta t$$

which we will simplify to

$$[q, p/m] = (\Delta x)^2/\Delta t.$$

taking q for the position x and p/m for velocity, the time derivative of position and ignoring the time shifting operator on the right hand side of the equation.

Understanding that Δt should be replaced by $i\Delta t$, and that, by comparison with the physics of a process at the Planck scale one can take

$$(\Delta x)^2/\Delta t = \hbar/m,$$

we have

$$[q, p/m] = (\Delta x)^2/i\Delta t = -i\hbar/m,$$

whence

$$[p, q] = i\hbar,$$

and we have arrived at Heisenberg's fundamental relationship between position and momentum. This mode of arrival is predicated on the recognition that only it represents a true interval of time. In the notion of time there is an inherent clock or an inherent shift of phase that is making a synchrony in our ability to observe, a precise dynamic beneath the apparent dynamic of the observed process. Once this substitution is made, once the correct imaginary value is placed in the temporal circuit, the patterns of quantum mechanics appear. In this way, quantum mechanics can be seen to emerge from the discrete.

The problem that we have examined in this section is the problem to understand the nature of quantum mechanics. In fact, we hope that the problem is seen to disappear the more we enter into the present viewpoint. A viewpoint is only on the periphery. The iterant from which the viewpoint emerges is in a superposition of indistinguishables, and can only be approached by varying the viewpoint until one is released from the particularities that a point of view contains.

3 Review of Non-Commutative Worlds

Now we begin the introduction to non-commutative worlds and a general discrete calculus. Our approach begins in an algebraic framework that naturally contains the formalism of the calculus, but not its notions of limits or constructions of spaces with specific locations, points and trajectories. Many patterns of physical law fit well into such an abstract framework. In this viewpoint one dispenses with continuum spacetime and replaces it by algebraic structure. Behind that structure, space stands ready to be constructed, by discrete derivatives and patterns of steps, or by starting with a discrete pattern in the form of a diagram, a network, a lattice, a knot, or a simplicial complex, and elaborating that structure until the specificity of spatio-temporal locations appear.

Poisson brackets allow one to connect classical notions of location with the non-commutative algebra used herein. Below the level of the Poisson brackets is a treatment of processes and operators as though they were variables in the same context as the variables in the classical calculus. In different degrees one lets go of the notion of classical variables and yet retains their form, as one makes a descent into the discrete. The discrete world of non-commutative operators is a world linked to our familiar world of continuous and commutative variables. This linkage is traditionally exploited in quantum mechanics to make the transition from the classical to the quantum. One can make the journey in the other direction, from the discrete and non-commutative to the “classical” and commutative, but that journey requires powers of invention and ingenuity that are the subject of this exploration. It is our conviction that the world is basically simple. To find simplicity in the complex requires special attention and care.

In starting from a discrete point of view one thinks of a sequence of states of the world S, S', S'', S''', \dots where S' denotes the state succeeding S in discrete time. It is natural to suppose that there is some measure of difference $DS^{(n)} = S^{(n+1)} - S^{(n)}$, and some way that states S and T might be combined to form a new state ST . We can thus think of world-states as operators in a non-commutative algebra with a temporal derivative $DS = S' - S$. At this bare level of the formalism the derivative does not satisfy the Leibniz rule. In fact it is easy to verify that $D(ST) = D(S)T + S'D(T)$. Remarkably, the Leibniz rule, and hence the formalisms of Newtonian calculus can be restored with the addition of one more operator J . In this instance J is a temporal shift operator with the property that $SJ = JS'$ for any state S . We then see that if $\nabla S = JD(S) = J(S' - S)$, then $\nabla(ST) = \nabla(S)T + S\nabla(T)$ for any states S and T . In fact $\nabla(S) = JS' - JS = SJ - JS = [S, J]$, so that this adjusted derivative is a commutator in the general calculus of states. This, in a nutshell, is our approach to non-commutative worlds. We begin with a very general framework that is a non-numerical calculus of states and operators. It is then fascinating and a topic of research to see how physics and mathematics fit into the frameworks so constructed.

A simplest and fundamental instance of these ideas is seen in the structure of $i = \sqrt{-1}$. We view i as an *iterant* (see Section 4), a discrete elementary dynamical system repeating in time the values $\{\dots -1, +1, -1, +1, \dots\}$. One can think of this system as resulting from the attempt to solve $i^2 = -1$ in the form $i = -1/i$. Then one iterates the transformation $x \rightarrow -1/x$ and

finds the oscillation from a starting value of $+1$ or -1 . In this sense i is identical in concept to a *primordial time*. Furthermore the algebraic structure of the complex numbers emerges from two conjugate views of this discrete series as $[-1, +1]$ and $[+1, -1]$. We introduce a temporal shift operator η such that $\eta[-1, +1] = [+1, -1]\eta$ and $\eta^2 = 1$ (sufficient to this purpose). Then we can define $i = [-1, +1]\eta$, endowing it with one view of the discrete oscillation and the sensitivity to shift the clock when interacting with itself or with another operator. See Sections 2 and 4 for the details of this reconstruction of the complex numbers. The point of the reconstruction for our purposes is that i becomes inextricably identified with elemental time, and so the physical substitution of it for t (Wick rotation) becomes, in this epistemology, an act of recognition of the nature of time.

Constructions are performed in a Lie algebra \mathcal{A} . One may take \mathcal{A} to be a specific matrix Lie algebra, or abstract Lie algebra. If \mathcal{A} is taken to be an abstract Lie algebra, then it is convenient to use the universal enveloping algebra so that the Lie product can be expressed as a commutator. In making general constructions of operators satisfying certain relations, it is understood that one can always begin with a free algebra and make a quotient algebra where the relations are satisfied.

On \mathcal{A} , a variant of calculus is built by defining derivations as commutators (or more generally as Lie products). For a fixed N in \mathcal{A} one defines

$$\nabla_N : \mathcal{A} \longrightarrow \mathcal{A}$$

by the formula

$$\nabla_N F = [F, N] = FN - NF.$$

∇_N is a derivation satisfying the Leibniz rule.

$$\nabla_N(FG) = \nabla_N(F)G + F\nabla_N(G).$$

Discrete Derivatives are Replaced by Commutators. There are many motivations for replacing derivatives by commutators. If $f(x)$ denotes (say) a function of a real variable x , and $\tilde{f}(x) = f(x + h)$ for a fixed increment h , define the *discrete derivative* Df by the formula $Df = (\tilde{f} - f)/h$, and find that the Leibniz rule is not satisfied. One has the basic formula for the discrete derivative of a product:

$$D(fg) = D(f)g + \tilde{f}D(g).$$

Correct this deviation from the Leibniz rule by introducing a new non-commutative operator J with the property that

$$fJ = J\tilde{f}.$$

Define a new discrete derivative in an extended non-commutative algebra by the formula

$$\nabla(f) = JD(f).$$

It follows at once that

$$\nabla(fg) = JD(f)g + J\tilde{f}D(g) = JD(f)g + fJD(g) = \nabla(f)g + f\nabla(g).$$

Note that

$$\nabla(f) = (J\tilde{f} - Jf)/h = (fJ - Jf)/h = [f, J/h].$$

In the extended algebra, discrete derivatives are represented by commutators, and satisfy the Leibniz rule. One can regard discrete calculus as a subset of non-commutative calculus based on commutators.

Advanced Calculus and Hamiltonian Mechanics or Quantum Mechanics in a Non-Commutative World. In \mathcal{A} there are as many derivations as there are elements of the algebra, and these derivations behave quite wildly with respect to one another. If one takes the concept of *curvature* as the non-commutation of derivations, then \mathcal{A} is a highly curved world indeed. Within \mathcal{A} one can build a tame world of derivations that mimics the behaviour of flat coordinates in Euclidean space. The description of the structure of \mathcal{A} with respect to these flat coordinates contains many of the equations and patterns of mathematical physics.

The flat coordinates Q^i satisfy the equations below with the P_j chosen to represent differentiation with respect to Q^j :

$$[Q^i, Q^j] = 0$$

$$[P_i, P_j] = 0$$

$$[Q^i, P_j] = \delta_{ij}.$$

Here δ_{ij} is the Kronecker delta, equal to 1 when $i = j$ and equal to 0 otherwise. Derivatives are represented by commutators.

$$\partial_i F = \partial F / \partial Q^i = [F, P_i],$$

$$\hat{\partial}_i F = \partial F / \partial P_i = [Q^i, F].$$

Our choice of commutators guarantees that the derivative of a variable with respect to itself is one and that the derivative of a variable with respect to a distinct variable is zero. Furthermore, the commuting of the variables with one another guarantees that mixed partial derivatives are independent of the order of differentiation. This is a flat non-commutative world.

Temporal derivative is represented by commutation with a special (Hamiltonian) element H of the algebra:

$$dF/dt = [F, H].$$

(For quantum mechanics, take $i\hbar dA/dt = [A, H]$.) These non-commutative coordinates are the simplest flat set of coordinates for description of temporal phenomena in a non-commutative world.

Hamilton's Equations are Part of the Mathematical Structure of Non-Commutative Advanced Calculus.

$$dP_i/dt = [P_i, H] = -[H, P_i] = -\partial H / \partial Q^i$$

$$dQ^i/dt = [Q^i, H] = \partial H / \partial P_i.$$

These are exactly Hamilton's equations of motion. The pattern of Hamilton's equations is built into the system.

The Simplest Time Series Leads to the Diffusion Constant and Heisenberg's Commuator.

Consider a time series $\{Q, Q', Q'', \dots\}$ with commuting scalar values. Let

$$\dot{Q} = \nabla Q = JDQ = J(Q' - Q)/\tau$$

where τ is an elementary time step (If Q denotes a times series value at time t , then Q' denotes the value of the series at time $t + \tau$). The shift operator J is defined by the equation $QJ = JQ'$ where this refers to any point in the time series so that $Q^{(n)}J = JQ^{(n+1)}$ for any non-negative integer n . Moving J across a variable from left to right, corresponds to one tick of the clock. This discrete, non-commutative time derivative satisfies the Leibniz rule.

This derivative ∇ also fits a significant pattern of discrete observation. Consider the act of observing Q at a given time and the act of observing (or obtaining) DQ at a given time. Since Q and Q' are ingredients in computing $(Q' - Q)/\tau$, the numerical value associated with DQ , it is necessary to let the clock tick once. Thus, if one first observe Q and then obtains DQ , the result is different (for the Q measurement) if one first obtains DQ , and then observes Q . In the second case, one finds the value Q' instead of the value Q , due to the tick of the clock.

1. Let $\dot{Q}Q$ denote the sequence: observe Q , then obtain \dot{Q} .
2. Let $Q\dot{Q}$ denote the sequence: obtain \dot{Q} , then observe Q .

The commutator $[Q, \dot{Q}]$ expresses the difference between these two orders of discrete measurement. In the simplest case, where the elements of the time series are commuting scalars, one has

$$[Q, \dot{Q}] = Q\dot{Q} - \dot{Q}Q = J(Q' - Q)^2/\tau.$$

Thus one can interpret the equation

$$[Q, \dot{Q}] = Jk$$

(k a constant scalar) as

$$(Q' - Q)^2/\tau = k.$$

This means that the process is a walk with spatial step

$$\Delta = \pm\sqrt{k\tau}$$

where k is a constant. In other words, one has the equation

$$k = \Delta^2/\tau.$$

This is the diffusion constant for a Brownian walk. A walk with spatial step size Δ and time step τ will satisfy the commutator equation above exactly when the square of the spatial step divided by the time step remains constant. This shows that the diffusion constant of a Brownian process is a structural property of that process, independent of considerations of probability and continuum limits.

Thus we can write (ignoring the timeshift operator J)

$$[Q, \dot{Q}] = (\Delta Q)^2/\tau.$$

If we work with physics at the Planck scale, then we can take τ as the Planck time and ΔQ as the Planck length. Then

$$(\Delta Q)^2/\tau = \hbar/m$$

where m is the Planck mass. However, we shall also Wick rotate the time from τ to $i\tau$ justifying $i\tau$ on the principle (described above) that τ should be multiplied by i to bring time into coincidence with an elemental time that is both a temporal operator (i) and a value (t). With this we obtain

$$[Q, \dot{Q}] = -i\hbar/m$$

or

$$[m\dot{Q}, Q] = i\hbar,$$

and taking $P = m\dot{Q}$, we have finally

$$[P, Q] = i\hbar.$$

Heisenberg's commutator for quantum mechanics is seen in the nexus of discrete physics and imaginary time.

Schroedinger's Equation is Discrete. Here is how the Heisenberg form of Schroedinger's equation fits in this context. Let $J = (1 + i\hbar H \Delta t)$. Then $\nabla\psi = [\psi, J/\Delta t]$, and we calculate

$$\nabla\psi = \psi[(1 + i\hbar H \Delta t)/\Delta t] - [(1 + i\hbar H \Delta t)/\Delta t]\psi = i\hbar[\psi, H].$$

This is exactly the form of the Heisenberg equation.

Dynamical Equations Generalize Gauge Theory and Curvature. One can take the general dynamical equation in the form

$$dQ^i/dt = \mathcal{G}_i$$

where $\{\mathcal{G}_1, \dots, \mathcal{G}_d\}$ is a collection of elements of \mathcal{A} . Write \mathcal{G}_i relative to the flat coordinates via $\mathcal{G}_i = P_i - A_i$. This is a definition of A_i and $\partial F/\partial Q^i = [F, P_i]$. The formalism of gauge theory appears naturally. In particular, if

$$\nabla_i(F) = [F, \mathcal{G}_i],$$

then one has the curvature

$$[\nabla_i, \nabla_j]F = [R_{ij}, F]$$

and

$$R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

This is the well-known formula for the curvature of a gauge connection. Aspects of geometry arise naturally in this context, including the Levi-Civita connection (which is seen as a consequence of the Jacobi identity in an appropriate non-commutative world).

One can consider the consequences of the commutator $[Q^i, \dot{Q}^j] = g_{ij}$, deriving that

$$\ddot{Q}^r = G_r + F_{rs}\dot{Q}^s + \Gamma_{rst}\dot{Q}^s\dot{Q}^t,$$

where G_r is the analogue of a scalar field, F_{rs} is the analogue of a gauge field and Γ_{rst} is the Levi-Civita connection associated with g_{ij} . This decomposition of the acceleration is uniquely determined by the given framework.

Non-commutative Electromagnetism and Gauge Theory. One can use this context to revisit the Feynman-Dyson derivation of electromagnetism from commutator equations, showing that most of the derivation is independent of any choice of commutators, but highly dependent upon the choice of definitions of the derivatives involved. Without any assumptions about initial commutator equations, but taking the right (in some sense simplest) definitions of the derivatives one obtains a significant generalization of the result of Feynman-Dyson. We give this derivation in Section 5 of the present paper, using diagrammatic algebra to clarify the structure. In this derivation we use X to denote the position vector rather than Q , as above.

Theorem With the appropriate [see below] definitions of the operators, and taking

$$\nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2, \quad H = \dot{X} \times \dot{X} \quad \text{and} \quad E = \partial_t \dot{X}, \quad \text{one has}$$

1. $\ddot{X} = E + \dot{X} \times H$
2. $\nabla \bullet H = 0$
3. $\partial_t H + \nabla \times E = H \times H$
4. $\partial_t E - \nabla \times H = (\partial_t^2 - \nabla^2)\dot{X}$

The key to the proof of this Theorem is the definition of the time derivative. This definition is as follows

$$\partial_t F = \dot{F} - \sum_i \dot{X}_i \partial_i(F) = \dot{F} - \sum_i \dot{X}_i [F, \dot{X}_i]$$

for all elements or vectors of elements F . The definition creates a distinction between space and time in the non-commutative world. A calculation reveals that

$$\ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X}).$$

This suggests taking $E = \partial_t \dot{X}$ as the electric field, and $B = \dot{X} \times \dot{X}$ as the magnetic field so that the Lorentz force law

$$\ddot{X} = E + \dot{X} \times B$$

is satisfied.

This result is applied to produce many discrete models of the Theorem. These models show that, just as the commutator $[X, \dot{X}] = Jk$ describes Brownian motion in one dimension, a generalization of electromagnetism describes the interaction of triples of time series in three dimensions.

Taking $\partial_t F = \dot{F} - \Sigma_i \dot{X}_i \partial_i(F) = \dot{F} - \Sigma_i \dot{X}_i [F, \dot{X}_i]$ as a definition of the partial derivative with respect to time is a natural move in this context because there is *no time variable* t in this non-commutative world. A formal move of this kind, matching a pattern from the commutative world to the mathematics of the non-commutative world is the theme of the Section 6 of this paper. In that section we consider the well known way to associate an operator to a product of commutative variables by taking a sum over all permutations of products of the operators corresponding to the individual variables. This provides a way to associate operator expressions with expressions in the commutative algebra, and hence to let a classical world correspond or map to a non-commutative world. To bind these worlds more closely, we can ask that the formulas for taking derivatives in the commutative world should have symmetrized operator product correspondences in the non-commutative world. In Section 6 we show how the resulting constraints are related to having a quadratic Hamiltonian (first order constraint) and to having a version of general relativity [3, 4] (second order constraint). Such constraints can be carried to all orders of derivatives, but the algebra of such constraints is, at the present time, in a very primitive state. We discuss some of the complexities of the constraint algebra in the Appendix to this paper.

In Section 7 we discuss the relationship of the Bianchi identity in non-commutative worlds and its role in the classical derivation of Einstein's equations. This suggests other avenues of relationship between general relativity and non-commutative worlds. The reader may well ask at this point if we propose quantum gravity via this framework. In our judgement, it is too early to tell.

Remark. While there is a large literature on non-commutative geometry, emanating from the idea of replacing a space by its ring of functions, work discussed herein is not written in that tradition. Non-commutative geometry does occur here, in the sense of geometry occurring in the context of non-commutative algebra. Derivations are represented by commutators. There are relationships between the present work and the traditional non-commutative geometry, but that is a subject for further exploration. In no way is this paper intended to be an introduction to that subject. The present summary is based on [15, 17, 18, 19, 20, 21, 22, 23, 24, 25] and the references cited therein.

The following references in relation to non-commutative calculus are useful in comparing with the present approach [2, 5, 7, 29]. Much of the present work is the fruit of a long series of discussions with Pierre Noyes. paper [27] also works with minimal coupling for the Feynman-Dyson derivation. The first remark about the minimal coupling occurs in the original paper by Dyson [1], in the context of Poisson brackets. The paper [8] is worth reading as a companion to Dyson. It is the purpose of this summary to indicate how non-commutative calculus can be used in foundations.

4 Iterants, Discrete Processes and Matrix Algebra

The primitive idea behind an iterant is a periodic time series or “waveform”

$$\cdots abababababab \cdots$$

The elements of the waveform can be any mathematically or empirically well-defined objects. We can regard the ordered pairs $[a, b]$ and $[b, a]$ as abbreviations for the waveform or as two points of view about the waveform (a first or b first). Call $[a, b]$ an *iterant*. One has the collection of transformations of the form $T[a, b] = [ka, k^{-1}b]$ leaving the product ab invariant. This tiny model contains the seeds of special relativity, and the iterants contain the seeds of general matrix algebra! Since this paper has been a combination of discussions of non-commutativity and time series, we include this appendix on iterants. A more complete discussion will appear elsewhere. For related discussion see [9, 10, 11, 12, 13, 14, 16, 28].

Define products and sums of iterants as follows

$$[a, b][c, d] = [ac, bd]$$

and

$$[a, b] + [c, d] = [a + c, b + d].$$

The operation of juxtaposition is multiplication while $+$ denotes ordinary addition in a category appropriate to these entities. These operations are natural with respect to the structural juxtaposition of iterants:

$$\cdots abababababab \cdots$$

$$\cdots cdcdcdcdcdcd \cdots$$

Structures combine at the points where they correspond. Waveforms combine at the times where they correspond. Iterants combine in juxtaposition.

If \bullet denotes any form of binary composition for the ingredients (a, b, \dots) of iterants, then we can extend \bullet to the iterants themselves by the definition $[a, b] \bullet [c, d] = [a \bullet c, b \bullet d]$. In this section we shall first apply this idea to Lorentz transformations, and then generalize it to other contexts.

So, to work: We have

$$[t - x, t + x] = [t, t] + [-x, x] = t[1, 1] + x[-1, 1].$$

Since $[1, 1][a, b] = [1a, 1b] = [a, b]$ and $[0, 0][a, b] = [0, 0]$, we shall write

$$1 = [1, 1]$$

and

$$0 = [0, 0].$$

Let

$$\sigma = [-1, 1].$$

σ is a significant iterant that we shall refer to as a *polarity*. Note that

$$\sigma\sigma = 1.$$

Note also that

$$[t - x, t + x] = t + x\sigma.$$

Thus the points of spacetime form an algebra analogous to the complex numbers whose elements are of the form $t + x\sigma$ with $\sigma\sigma = 1$ so that

$$(t + x\sigma)(t' + x'\sigma) = tt' + xx' + (tx' + xt')\sigma.$$

In the case of the Lorentz transformation it is easy to see the elements of the form $[k, k^{-1}]$ translate into elements of the form

$$T(v) = [(1 + v)/\sqrt{(1 - v^2)}, (1 - v)/\sqrt{(1 - v^2)}] = [k, k^{-1}].$$

Further analysis shows that v is the relative velocity of the two reference frames in the physical context. Multiplication now yields the usual form of the Lorentz transform

$$\begin{aligned} T_k(t + x\sigma) &= T(v)(t + x\sigma) \\ &= (1/\sqrt{(1 - v^2)} - v\sigma/\sqrt{(1 - v^2)})(t + x\sigma) \\ &= (t - xv)/\sqrt{(1 - v^2)} + (x - vt)\sigma/\sqrt{(1 - v^2)} \\ &= t' + x'\sigma. \end{aligned}$$

The algebra that underlies this iterant presentation of special relativity is a relative of the complex numbers with a special element σ of square one rather than minus one ($i^2 = -1$ in the complex numbers).

The appearance of a square root of minus one unfolds naturally from iterant considerations. Define the “shift” operator η on iterants by the equation

$$\eta[a, b] = [b, a]\eta$$

with $\eta^2 = 1$. Sometimes it is convenient to think of η as a delay operator, since it shifts the waveform $\dots ababab\dots$ by one internal time step. Now define

$$i = [-1, 1]\eta$$

We see at once that

$$ii = [-1, 1]\eta[-1, 1]\eta = [-1, 1][1, -1]\eta^2 = [-1, 1][1, -1] = [-1, -1] = -1.$$

Thus

$$ii = -1.$$

Here we have described i in a *new* way as the superposition of the waveform $\epsilon = [-1, 1]$ and the temporal shift operator η . By writing $i = \epsilon\eta$ we recognize an active version of the waveform that shifts temporally when it is observed. This theme of including the result of time in observations of a discrete system occurs at the foundation of our construction.

4.1 MATRIX ALGEBRA VIA ITERANTS

Matrix algebra has some strange wisdom built into its very bones. Consider a two dimensional periodic pattern or “waveform.”

.....
 ...ababababababab...
 ...cdcdcdcdcdcdcd...
 ...ababababababab...
 ...cdcdcdcdcdcdcd...
 ...ababababababab...

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} b & a \\ d & c \end{pmatrix}, \begin{pmatrix} c & d \\ a & b \end{pmatrix}, \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

Above are some of the matrices apparent in this array. Compare the matrix with the “two dimensional waveform” shown above. A given matrix freezes out a way to view the infinite waveform. In order to keep track of this patterning, lets write

$$[a, d] + [b, c]\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

where

$$[x, y] = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

and

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The four matrices that can be framed in the two-dimensional wave form are all obtained from the two iterants $[a, d]$ and $[b, c]$ via the shift operation $\eta[x, y] = [y, x]\eta$ which we shall denote by an overbar as shown below

$$\overline{[x, y]} = [y, x].$$

Letting $A = [a, d]$ and $B = [b, c]$, we see that the four matrices seen in the grid are

$$A + B\eta, B + A\eta, \overline{B} + \overline{A}\eta, \overline{A} + \overline{B}\eta.$$

The operator η has the effect of rotating an iterant by ninety degrees in the formal plane. Ordinary matrix multiplication can be written in a concise form using the following rules:

$$\begin{aligned}\eta\eta &= 1 \\ \eta Q &= \overline{Q}\eta\end{aligned}$$

where Q is any two element iterant.

For example, let $\epsilon = [-1, 1]$ so that $\bar{\epsilon} = -\epsilon$ and $\epsilon\epsilon = [1, 1] = 1$. Let

$$i = \epsilon\eta.$$

Then

$$ii = \epsilon\eta\epsilon\eta = \epsilon\bar{\epsilon}\eta\eta = \epsilon(-\epsilon) = -\epsilon\epsilon = -1.$$

We have reconstructed the square root of minus one in the form of the matrix

$$i = \epsilon\eta = [-1, 1]\eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

More generally, we see that

$$(A + B\eta)(C + D\eta) = (AC + B\overline{D}) + (AD + B\overline{C})\eta$$

writing the 2×2 matrix algebra as a system of hypercomplex numbers. Note that

$$(A + B\eta)(\overline{A} - B\eta) = A\overline{A} - B\overline{B}$$

The formula on the right corresponds to the determinant of the matrix. Thus we define the *conjugate* of $A + B\eta$ by the formula

$$\overline{A + B\eta} = \overline{A} - B\eta.$$

These patterns generalize to higher dimensional matrix algebra.

It is worth pointing out the first precursor to the quaternions: This precursor is the system

$$\{\pm 1, \pm \epsilon, \pm \eta, \pm i\}.$$

Here $\epsilon\epsilon = 1 = \eta\eta$ while $i = \epsilon\eta$ so that $ii = -1$. The basic operations in this algebra are those of epsilon and eta. Eta is the delay shift operator that reverses the components of the iterant. Epsilon negates one of the components, and leaves the order unchanged. The quaternions arise directly from these two operations once we construct an extra square root of minus one that commutes with them. Call this extra root of minus one $\sqrt{-1}$. Then the quaternions are generated by

$$\{i = \epsilon\eta, j = \sqrt{-1}\bar{\epsilon}, k = \sqrt{-1}\eta\}$$

with

$$i^2 = j^2 = k^2 = ijk = -1.$$

The “right” way to generate the quaternions is to start at the bottom iterant level with boolean values of 0 and 1 and the operation EXOR (exclusive or). Build iterants on this, and matrix algebra from these iterants. This gives the square root of negation. Now take pairs of values from this new algebra and build 2×2 matrices again. The coefficients include square roots of negation that commute with constructions at the next level and so quaternions appear in the third level of this hierarchy.

4.2 Matrix Algebra in General

Construction of matrix algebra in general proceeds as follows. Let M be an $n \times n$ matrix over a ring R . Let $M = (m_{ij})$ denote the matrix entries. Let π be an element of the symmetric group S_n so that $\pi_1, \pi_2, \dots, \pi_n$ is a permutation of $1, 2, \dots, n$. Let $v = (v_1, v_2, \dots, v_n)$ denote a vector with these components. Let $\Delta(v)$ denote the diagonal matrix whose i -th diagonal entry is v_i . Let $v^\pi = (v_{\pi_1}, \dots, v_{\pi_n})$. Let $\Delta^\pi(v) = \Delta(v^\pi)$. Let Δ denote any diagonal matrix and Δ^π denote the corresponding permuted diagonal matrix as just described. Let $[\pi]$ denote the permutation matrix obtained by taking the i -th row of $[\pi]$ to be the π_i -th row of the identity matrix. Note that $[\pi]\Delta = \Delta^\pi[\pi]$. For each element π of S_n define the vector $v(M, \pi) = (m_{1\pi_1}, \dots, m_{n\pi_n})$ and the diagonal matrix $\Delta[M]_\pi = \Delta(v(M, \pi))$.

Theorem. $M = (1/(n-1!)) \sum_{\pi \in S_n} \Delta[M]_\pi [\pi]$.

The proof of this theorem is omitted here. Note that the theorem expresses any square matrix as a sum of products of diagonal matrices and permutation matrices. Diagonal matrices add and multiply by adding and multiplying their corresponding entries. They are acted upon by permutations as described above. This means that any matrix algebra can be embedded in an algebra that has the structure of a group ring of the permutation group with coefficients Δ in an algebra (here the diagonal matrices) that are acted upon by the permutation group, and following the rule $[\pi]\Delta = \Delta^\pi[\pi]$. This is a full generalization of the case $n = 2$ described in the last section.

For example, we have the following expansion of a 3×3 matrix:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \frac{1}{2!} \left[\begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & k \end{pmatrix} + \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & f \\ g & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & c \\ d & 0 & 0 \\ 0 & h & 0 \end{pmatrix} + \right. \\ \left. \begin{pmatrix} 0 & 0 & c \\ 0 & e & 0 \\ g & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b & 0 \\ d & 0 & 0 \\ 0 & 0 & k \end{pmatrix} + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & f \\ 0 & h & 0 \end{pmatrix} \right].$$

Here, each term factors as a diagonal matrix multiplied by a permutation matrix as in

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & f \\ 0 & h & 0 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & h \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is amusing to note that this theorem tells us that up to the factor of $1/(n-1)!$ a unitary matrix that has unit complex numbers as its entries is a sum of simpler unitary transformations factored into diagonal and permutation matrices. In quantum computing parlance, such a unitary matrix is a sum of products of phase gates and products of swap gates (since each permutation is a product of transpositions).

A reason for discussing these formulations of matrix algebra in the present context is that one sees that matrix algebra is generated by the simple operations of juxtaposed addition and multiplication, and by the use of permutations as operators. These are unavoidable discrete elements, and so the operations of matrix algebra can be motivated on the basis of discrete physical ideas and non-commutativity. The richness of continuum formulations, infinite matrix algebra, and symmetry grows naturally out of finite matrix algebra and hence out of the discrete.

5 Generalized Feynman Dyson Derivation

In this section we assume that specific time-varying coordinate elements X_1, X_2, X_3 of the algebra \mathcal{A} are given. *We do not assume any commutation relations about X_1, X_2, X_3 .*

In this section we no longer avail ourselves of the commutation relations that are in back of the original Feynman-Dyson derivation. We do take the definitions of the derivations from that previous context. Surprisingly, the result is very similar to the one of Feynman and Dyson, as we shall see.

Here $A \times B$ is the non-commutative vector cross product:

$$(A \times B)_k = \sum_{i,j=1}^3 \epsilon_{ijk} A_i B_j.$$

(We will drop this summation sign for vector cross products from now on.) Then, with $B = \dot{X} \times \dot{X}$, we have

$$B_k = \epsilon_{ijk} \dot{X}_i \dot{X}_j = (1/2) \epsilon_{ijk} [\dot{X}_i, \dot{X}_j].$$

The epsilon tensor ϵ_{ijk} is defined for the indices $\{i, j, k\}$ ranging from 1 to 3, and is equal to 0 if there is a repeated index and is otherwise equal to the sign of the permutation of 123 given by ijk . We represent dot products and cross products in diagrammatic tensor notation as indicated in Figure 2 and Figure 3. In Figure 2 we indicate the epsilon tensor by a trivalent vertex. The indices of the tensor correspond to labels for the three edges that impinge on the vertex. The diagram is drawn in the plane, and is well-defined since the epsilon tensor is invariant under cyclic permutation of its indices.

We will define the fields E and B by the equations

$$B = \dot{X} \times \dot{X} \quad \text{and} \quad E = \partial_t \dot{X}.$$

We will see that E and B obey a generalization of the Maxwell Equations, and that this generalization describes specific discrete models. The reader should note that this means that a significant part of the *form* of electromagnetism is the consequence of choosing three coordinates of space, and the definitions of spatial and temporal derivatives with respect to them. The background process that is being described is otherwise arbitrary, and yet appears to obey physical laws once these choices are made.

In this section we will use diagrammatic matrix methods to carry out the mathematics. In general, in a diagram for matrix or tensor composition, we sum over all indices labeling any edge in the diagram that has no free ends. Thus matrix multiplication corresponds to the connecting of edges between diagrams, and to the summation over common indices. With this interpretation of compositions, view the first identity in Figure 2. This is a fundamental identity about the epsilon, and corresponds to the following lemma.

$$\begin{aligned}
 & \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \quad \quad c \end{array} = \epsilon_{abc} \\
 & \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \quad \quad i \\ \diagup \quad \diagdown \\ d \quad c \end{array} = - \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \quad \quad d \\ \diagup \quad \diagdown \\ c \quad c \end{array} \\
 & \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \quad \quad c \end{array} = - \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \quad \quad c \end{array} + \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \quad \quad c \end{array} \\
 & \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \quad \quad c \end{array} = \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \quad \quad c \end{array} - \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \quad \quad c \end{array} \\
 & \boxed{\begin{array}{c} A \quad B \quad C \\ \diagdown \quad \diagup \\ \quad \quad C \end{array} = \begin{array}{c} A \quad B \quad C \\ \diagdown \quad \diagup \\ \quad \quad C \end{array} - \begin{array}{c} A \quad B \quad C \\ \diagdown \quad \diagup \\ \quad \quad C \end{array}}
 \end{aligned}$$

Figure 2: **Epsilon Identity**

Lemma. (View Figure 2) Let ϵ_{ijk} be the epsilon tensor taking values 0, 1 and -1 as follows: When ijk is a permutation of 123, then ϵ_{ijk} is equal to the sign of the permutation. When ijk

contains a repetition from $\{1, 2, 3\}$, then the value of epsilon is zero. Then ϵ satisfies the labeled identity in Figure 2 in terms of the Kronecker delta.

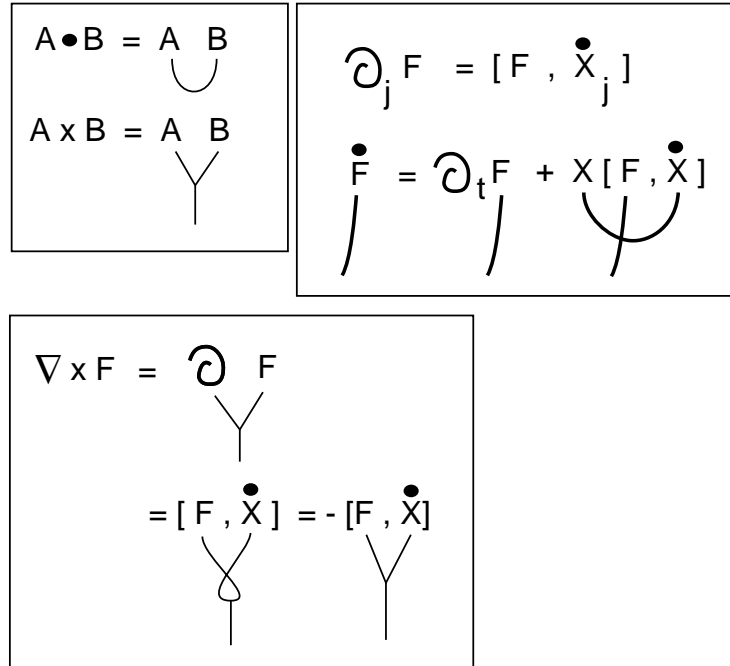
$$\sum_i \epsilon_{abi} \epsilon_{cdi} = -\delta_{ad} \delta_{bc} + \delta_{ac} \delta_{bd}.$$

The proof of this identity is left to the reader. The identity itself will be referred to as the *epsilon identity*. The epsilon identity is a key structure in the work of this section, and indeed in all formulas involving the vector cross product.

The reader should compare the formula in this Lemma with the diagrams in Figure 2. The first two diagram are two versions of the Lemma. In the third diagram the labels are capitalized and refer to vectors A, B and C . We then see that the epsilon identity becomes the formula

$$A \times (B \times C) = (A \bullet C)B - (A \bullet B)C$$

for vectors in three-dimensional space (with commuting coordinates, and a generalization of this identity to our non-commutative context. Refer to Figure 3 for the diagrammatic definitions of dot and cross product of vectors. We take these definitions (with implicit order of multiplication) in the non-commutative context.



Remarks on the Derivatives.

1. Since we do not assume that $[X_i, \dot{X}_j] = \delta_{ij}$, nor do we assume $[X_i, X_j] = 0$, it will not follow that E and B commute with the X_i .

2. We define

$$\partial_i(F) = [F, \dot{X}_i],$$

and the reader should note that, these spatial derivations are no longer flat in the sense of section 1 (nor were they in the original Feynman-Dyson derivation). See Figure 3 for the diagrammatic version of this definition.

3. We define $\partial_t = \partial/\partial t$ by the equation

$$\partial_t F = \dot{F} - \sum_i \dot{X}_i \partial_i(F) = \dot{F} - \sum_i \dot{X}_i [F, \dot{X}_i]$$

for all elements or vectors of elements F . We take this equation as the global definition of the temporal partial derivative, even for elements that are not commuting with the X_i . This notion of temporal partial derivative ∂_t is a least relation that we can write to describe the temporal relationship of an arbitrary non-commutative vector F and the non-commutative coordinate vector X . See Figure 3 for the diagrammatic version of this definition.

4. In defining

$$\partial_t F = \dot{F} - \sum_i \dot{X}_i \partial_i(F),$$

we are using the definition itself to obtain a notion of the variation of F with respect to time. The definition itself creates a distinction between space and time in the non-commutative world.

5. The reader will have no difficulty verifying the following formula:

$$\partial_t(FG) = \partial_t(F)G + F\partial_t(G) + \sum_i \partial_i(F)\partial_i(G).$$

This formula shows that ∂_t does not satisfy the Leibniz rule in our non-commutative context. This is true for the original Feynman-Dyson context, and for our generalization of it. All derivations in this theory that are defined directly as commutators do satisfy the Leibniz rule. Thus ∂_t is an operator in our theory that does not have a representation as a commutator.

6. We define divergence and curl by the equations

$$\nabla \bullet B = \sum_{i=1}^3 \partial_i(B_i)$$

and

$$(\nabla \times E)_k = \epsilon_{ijk} \partial_i(E_j).$$

See Figure 3 and Figure 5 for the diagrammatic versions of curl and divergence.

Now view Figure 4. We see from this Figure that it follows directly from the definition of the time derivatives (as discussed above) that

$$\ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X}).$$

This is our motivation for defining

$$E = \partial_t \dot{X}$$

and

$$B = \dot{X} \times \dot{X}.$$

With these definition in place we have

$$\ddot{X} = E + \dot{X} \times B,$$

giving an analog of the Lorentz force law for this theory.

Just for the record, look at the following algebraic calculation for this derivative:

$$\begin{aligned} \dot{F} &= \partial_t F + \Sigma_i \dot{X}_i [F, \dot{X}_i] \\ &= \partial_t F + \Sigma_i (\dot{X}_i F \dot{X}_i - \dot{X}_i \dot{X}_i F) \\ &= \partial_t F + \Sigma_i (\dot{X}_i F \dot{X}_i - \dot{X}_i F_i \dot{X}) + \dot{X}_i F_i \dot{X} - \dot{X}_i \dot{X}_i F \end{aligned}$$

Hence

$$\dot{F} = \partial_t F + \dot{X} \times F + (\dot{X} \bullet F) \dot{X} - (\dot{X} \bullet \dot{X}) F$$

(using the epsilon identity). Thus we have

$$\ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X}) + (\dot{X} \bullet \dot{X}) \dot{X} - (\dot{X} \bullet \dot{X}) \dot{X},$$

whence

$$\ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X}).$$

In Figure 5, we give the derivation that B has zero divergence.

Figures 6 and 7 compute derivatives of B and the Curl of E , culminating in the formula

$$\partial_t B + \nabla \times E = B \times B.$$

In classical electromagnetism, there is no term $B \times B$. This term is an artifact of our non-commutative context. In discrete models, as we shall see at the end of this section, there is no escaping the effects of this term.

Finally, Figure 8 gives the diagrammatic proof that

$$\partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2) \dot{X}.$$

This completes the proof of the Theorem below.

$$\begin{aligned}
\dot{\vec{F}} &= \partial_t \vec{F} + \vec{X} [\vec{F}, \vec{X}] \\
\dot{\vec{F}} &= \partial_t \vec{F} + \vec{X} \vec{F} \vec{X} - \vec{X} \vec{X} \vec{F} \\
\ddot{\vec{X}} &= \partial_t \dot{\vec{X}} + \dot{\vec{X}} \dot{\vec{X}} \dot{\vec{X}} - \dot{\vec{X}} \dot{\vec{X}} \dot{\vec{X}} \\
&= \partial_t \dot{\vec{X}} + \dot{\vec{X}} \dot{\vec{X}} \dot{\vec{X}} \\
\boxed{\ddot{\vec{X}} = \partial_t \dot{\vec{X}} + \dot{\vec{X}} \times (\dot{\vec{X}} \times \dot{\vec{X}})}
\end{aligned}$$

Figure 4: **The Formula for Acceleration**

Electromagnetic Theorem With the above definitions of the operators, and taking

$$\nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2, \quad B = \dot{\vec{X}} \times \dot{\vec{X}} \quad \text{and} \quad E = \partial_t \dot{\vec{X}} \quad \text{we have}$$

1. $\ddot{\vec{X}} = E + \dot{\vec{X}} \times B$
2. $\nabla \bullet B = 0$
3. $\partial_t B + \nabla \times E = B \times B$
4. $\partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2) \dot{\vec{X}}$

Remark. Note that this Theorem is a non-trivial generalization of the Feynman-Dyson derivation of electromagnetic equations. In the Feynman-Dyson case, one assumes that the commutation relations

$$[X_i, X_j] = 0$$

and

$$[X_i, \dot{X}_j] = \delta_{ij}$$

are given, *and* that the principle of commutativity is assumed, so that if A and B commute with the X_i then A and B commute with each other. One then can interpret ∂_i as a standard derivative with $\partial_i(X_j) = \delta_{ij}$. Furthermore, one can verify that E_j and B_j both commute with the X_i . From

$$\begin{aligned}
E &= \partial_t \dot{X} & B &= \dot{X} \times \dot{X} \\
\dot{X} &= E + \dot{X} \times B \\
\nabla \cdot B &= [\dot{B}, \dot{X}] \\
&= \dot{B} \dot{X} - \dot{X} \dot{B} = \dot{X} \dot{X} \dot{X} - \dot{X} \dot{X} \dot{X} = 0 \\
\nabla \cdot B &= 0
\end{aligned}$$

Figure 5: **Divergence of B**

this it follows that $\partial_t(E)$ and $\partial_t(B)$ have standard interpretations and that $B \times B = 0$. The above formulation of the Theorem adds the description of E as $\partial_t(\dot{X})$, a non-standard use of ∂_t in the original context of Feynman-Dyson, where ∂_t would only be defined for those A that commute with X_i . In the same vein, the last formula $\partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2)\dot{X}$ gives a way to express the remaining Maxwell Equation in the Feynman-Dyson context.

Remark. Note the role played by the epsilon tensor ϵ_{ijk} throughout the construction of generalized electromagnetism in this section. The epsilon tensor is the structure constant for the Lie algebra of the rotation group $SO(3)$. If we replace the epsilon tensor by a structure constant f_{ijk} for a Lie algebra \mathcal{G} of dimension d such that the tensor is invariant under cyclic permutation ($f_{ijk} = f_{kij}$), then most of the work in this section will go over to that context. We would then have d operator/variables X_1, \dots, X_d and a generalized cross product defined on vectors of length d by the equation

$$(A \times B)_k = f_{ijk} A_i B_j.$$

The Jacobi identity for the Lie algebra \mathcal{G} implies that this cross product will satisfy

$$A \times (B \times C) = (A \times B) \times C + [B \times (A \times C)]$$

where

$$([B \times (A \times C)]_r = f_{klr} f_{ijk} A_i B_k C_j.$$

This extension of the Jacobi identity holds as well for the case of non-commutative cross product defined by the epsilon tensor. It is therefore of interest to explore the structure of generalized non-commutative electromagnetism over other Lie algebras (in the above sense). This will be the subject of another paper.

$$\mathcal{D}_t B = \dot{B} + \dot{X} [\dot{X}, B]$$

$$\begin{aligned} \dot{B} &= (1/2) [\dot{X}, \dot{X}] = [\ddot{X}, \dot{X}] \\ &= [E, \dot{X}] + [\dot{X} \times B, \dot{X}] \\ &= -\nabla \times E + [\dot{X} B, \dot{X}] \end{aligned}$$

Figure 6: **Computing \dot{B}**

5.1 Discrete Thoughts

In the hypotheses of the Electromagnetic Theorem, we are free to take any non-commutative world, and the Electromagnetic Theorem will be satisfied in that world. For example, we can take each X_i to be an arbitrary time series of real or complex numbers, or bitstrings of zeroes and ones. The global time derivative is defined by

$$\dot{F} = J(F' - F) = [F, J],$$

where $FJ = JF'$. This is the non-commutative discrete context discussed in sections 2 and 3. We will write

$$\dot{F} = J\Delta(F)$$

where $\Delta(F)$ denotes the classical discrete derivative

$$\Delta(F) = F' - F.$$

With this interpretation X is a vector with three real or complex coordinates at each time, and

$$B = \dot{X} \times \dot{X} = J^2 \Delta(X') \times \Delta(X)$$

while

$$E = \ddot{X} - \dot{X} \times (\dot{X} \times \dot{X}) = J^2 \Delta^2(X) - J^3 \Delta(X'') \times (\Delta(X') \times \Delta(X)).$$

Note how the non-commutative vector cross products are composed through time shifts in this context of temporal sequences of scalars. The advantage of the generalization now becomes apparent. We can create very simple models of generalized electromagnetism with only the simplest of discrete materials. In the case of the model in terms of triples of time series, the

$$\begin{aligned}
\partial_t B + \nabla \times E &= \dot{X} [\dot{X}, B] + [\dot{X} B, \dot{X}] \\
&= \dot{X} [\dot{X}, B] + [\dot{X} B, \dot{X}] + [\dot{X} B, \dot{X}] \\
&= -\dot{X} \dot{X} B + \dot{X} \dot{X} B \quad (\text{Note that } \dot{X} B = B \dot{X}) \\
&= \dot{X} \dot{X} B = B \times B \\
\boxed{\partial_t B + \nabla \times E = B \times B}
\end{aligned}$$

Figure 7: **Curl of E**

generalized electromagnetic theory is a theory of measurements of the time series whose key quantities are

$$\Delta(X') \times \Delta(X)$$

and

$$\Delta(X'') \times (\Delta(X') \times \Delta(X)).$$

It is worth noting the forms of the basic derivations in this model. We have, assuming that F is a commuting scalar (or vector of scalars) and taking $\Delta_i = X'_i - X_i$,

$$\partial_i(F) = [F, \dot{X}_i] = [F, J\Delta_i] = FJ\Delta_i - J\Delta_i F = J(F'\Delta_i - \Delta_i F) = \dot{F}\Delta_i$$

and for the temporal derivative we have

$$\partial_t F = J[1 - J\Delta' \bullet \Delta]\Delta(F)$$

where $\Delta = (\Delta_1, \Delta_2, \Delta_3)$.

6 Constraints - Classical Physics and General Relativity

The program here is to investigate restrictions in a non-commutative world that are imposed by asking for a specific correspondence between classical variables acting in the usual context of continuum calculus, and non-commutative operators corresponding to these classical variables.

$E = \partial_t \dot{X} \longrightarrow \partial_t E = \partial_t^2 \dot{X}$
$\begin{aligned} \nabla \times B &= \partial \dot{X} \dot{X} \\ &= -\partial \dot{X} \dot{X} + \partial \dot{X} \dot{X} \\ &= \partial [\dot{X}, \dot{X}] = \{\partial \partial\} \dot{X} = \nabla^2 \dot{X} \end{aligned}$
$\partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2) \dot{X}$

Figure 8: **Curl of B**

By asking for the simplest constraints we find the need for a quadratic Hamiltonian and a remarkable relationship with Einstein's equations for general relativity [3, 4]. There is a hierarchy of constraints of which we only analyze the first two levels. An appendix to this paper indicates a direction for exploring the algebra of the higher constraints.

If, for example, we let x and y be classical variables and X and Y the corresponding non-commutative operators, then we ask that x^n correspond to X^n and that y^n correspond to Y^n for positive integers n . We further ask that linear combinations of classical variables correspond to linear combinations of the corresponding operators. These restrictions tell us what happens to products. For example, we have classically that $(x + y)^2 = x^2 + 2xy + y^2$. This, in turn must correspond to $(X + Y)^2 = X^2 + XY + YX + Y^2$. From this it follows that $2xy$ corresponds to $XY + YX$. Hence xy corresponds to

$$\{XY\} = (XY + YX)/2.$$

By a similar calculation, if x_1, x_2, \dots, x_n are classical variables, then the product $x_1 x_2 \dots x_n$ corresponds to

$$\{X_1 X_2 \dots X_n\} = (1/n!) \sum_{\sigma \in S_n} X_{\sigma_1} X_{\sigma_2} \dots X_{\sigma_n}.$$

where S_n denotes all permutations of $1, 2, \dots, n$. Note that we use curly brackets for these symmetrizers and square brackets for commutators as in $[A, B] = AB - BA$.

We can formulate constraints in the non-commutative world by asking for a correspondence between familiar differentiation formulas in continuum calculus and the corresponding formulas in the non-commutative calculus, where all derivatives are expressed via commutators. We will detail how this constraint algebra works in the first few cases. Exploration of these constraints

has been pioneered by Tony Deakin [3, 4]. The author of this paper and Tony Deakin are planning a comprehensive paper on the consequences of these constraints in the interface between classical and quantum mechanics.

Recall that the temporal derivative in a non-commutative world is represented by commutator with an operator H that can be interpreted as the Hamiltonian operator in certain contexts.

$$\dot{\Theta} = [\Theta, H].$$

For this discussion, we shall take a collection Q^1, Q^2, \dots, Q^n of operators to represent spatial coordinates q^1, q^2, \dots, q^n . The Q^i commute with one another, and the derivatives with respect to Q^i are represented by operators P^i so that

$$\partial\Theta/\partial Q^i = \Theta_i = [\Theta, P^i].$$

We also write

$$\partial\Theta/\partial P^i = \Theta^i = [Q^i, \Theta].$$

To this purpose, we assume that $[Q^i, P^j] = \delta^{ij}$ and that the P^j commute with one another (so that mixed partial derivatives with respect to the Q^i are independent of order of differentiation).

Note that

$$\dot{Q}^i = [Q^i, H] = H^i.$$

It will be convenient for us to write H^i in place of \dot{Q}^i in the calculations to follow.

The First Constraint. The *first constraint* is the equation

$$\dot{\Theta} = \{\dot{Q}^i \Theta_i\} = \{H^i \Theta_i\}.$$

This equation expresses the symmetrized version of the usual calculus formula $\dot{\theta} = \dot{q}^i \theta_i$. It is worth noting that the first constraint is satisfied by the quadratic Hamiltonian

$$H = \frac{1}{4}(g_{ij} P^i P^j + P^i P^j g_{ij})$$

where $g_{ij} = g_{ji}$ and the g_{ij} commute with the Q^k . We leave the verification of this point to the reader, and note that the fact that the quadratic Hamiltonian does satisfy the first constraint shows how the constraints bind properties of classical physics (in this case Hamiltonian mechanics) to the non-commutative world.

The Second Constraint. The *second constraint* is the symmetrized analog of the second temporal derivative:

$$\ddot{\Theta} = \{\dot{H}^i \Theta_i\} + \{H^i H^j \Theta_{ij}\}.$$

However, by differentiating the first constraint we have

$$\ddot{\Theta} = \{\dot{H}^i \Theta_i\} + \{H^i \{H^j \Theta_{ij}\}\}$$

Thus the second constraint is equivalent to the equation

$$\{H^i \{H^j \Theta_{ij}\}\} = \{H^i H^j \Theta_{ij}\}.$$

We now reformulate this version of the constraint in the following theorem.

Theorem. The second constraint in the form $\{H^i\{H^j\Theta_{ij}\}\} = \{H^iH^j\Theta_{ij}\}$ is equivalent to the equation

$$[[\Theta_{ij}, H^j], H^i] = 0.$$

Proof. We can shortcut the calculations involved in proving this Theorem by looking at the properties of symbols A, B, C such that $AB = BA$, $ACB = BCA$. Formally these mimic the behaviour of $A = H^i, B = H^j, C = \Theta_{ij}$ in the expressions $H^iH^j\Theta_{ij}$ and $H^i\Theta_{ij}H^j$ since $\Theta_{ij} = \Theta_{ji}$, and the Einstein summation convention is in place. Then

$$\begin{aligned}\{A\{BC\}\} &= \frac{1}{4}(A(BC + CB) + (BC + CB)A) \\ &= \frac{1}{4}(ABC + ACB + BCA + CBA), \\ \{ABC\} &= \frac{1}{6}(ABC + ACB + BAC + BCA + CAB + CBA).\end{aligned}$$

So

$$\begin{aligned}\{ABC\} - \{A\{BC\}\} &= \frac{1}{12}(-ABC - ACB + 2BAC - BCA + 2CAB - CBA) \\ &= \frac{1}{12}(ABC - 2ACB + CAB) \\ &= \frac{1}{12}(ABC - 2BCA + CBA) \\ &= \frac{1}{12}(A(BC - CB) + (CB - BC)A) \\ &= \frac{1}{12}(A[B, C] - [B, C]A) \\ &= \frac{1}{12}[A, [B, C]].\end{aligned}$$

Thus the second constraint is equivalent to the equation

$$[H^i, [H^j, \Theta_{ij}]] = 0.$$

This in turn is equivalent to the equation

$$[[\Theta_{ij}, H^j], H^i] = 0,$$

completing the proof of the Theorem.

Remark. If we define

$$\nabla^i(\Theta) = [\Theta, H^i] = [\Theta, \dot{Q}^i]$$

then this is the natural covariant derivative that was described in the introduction to this paper. Thus the second order constraint is

$$\nabla^i(\nabla^j(\Theta_{ij})) = 0.$$

Note that

$$\begin{aligned} \nabla^i(\nabla^j(\Theta_{ij})) &= [[\Theta_{ij}, H^j], H^i] \\ &= -[[H^i, \Theta_{ij}], H^j] - [[H^j, H^i], \Theta_{ij}] \\ &= \nabla^j(\nabla^i(\Theta_{ij})) + [[H^i, H^j], \Theta_{ij}] \\ &= \nabla^i(\nabla^j(\Theta_{ij})) + [[H^i, H^j], \Theta_{ij}]. \end{aligned}$$

Hence the second order constraint is equivalent to the equation

$$[[H^i, H^j], \Theta_{ij}] = 0.$$

This equation weaves the curvature of ∇ with the flat derivatives of Θ .

A Relationship with General Relativity. Again, if we define

$$\nabla^i(\Theta) = [\Theta, H^i] = [\Theta, \dot{Q}^i]$$

then this is the natural covariant derivative that was described in the introduction to this paper. Thus the second order constraint is

$$\nabla^i(\nabla^j(\Theta_{ij})) = 0.$$

If we use the quadratic Hamiltonian $H = \frac{1}{4}(g_{ij}P^iP^j + P^iP^jg_{ij})$ as above, then with $\Theta = g^{lm}$ the second constraint becomes the equation

$$g^{uv}(g^{jk}g^{lm}_{,jku})_v = 0.$$

Deakin and Kilmister [4] observe that this last equation specializes to a fourth order version of Einstein's field equation for vacuum general relativity:

$$K_{ab} = g^{ef}(R_{ab;ef} + \frac{2}{3}R_{ae}R_{fb}) = 0$$

where $a, b, e, f = 1, 2, \dots, n$ and R is the curvature tensor corresponding to the metric g_{ab} . This equation has been studied by Deakin in [4]. It remains to be seen what the consequences for general relativity are in relation to this formulation, and it remains to be seen what the further consequences of higher order constraints will be.

The algebra of the higher order constraints is under investigation at this time.

7 On the Algebra of Constraints

We have the usual advanced calculus formula $\dot{\theta} = \dot{q}^i \theta_i$. We shall define $h^j = \dot{q}^j$ so that we can write $\dot{\theta} = h^i \theta_i$. We can then calculate successive derivatives with $\theta^{(n)}$ denoting the n -th temporal derivative of θ .

$$\begin{aligned}\theta^{(1)} &= h^i \theta_i \\ \theta^{(2)} &= h^{i(1)} \theta_i + h^i h^j \theta_{ij} \\ \theta^{(3)} &= h^{i(2)} \theta_i + 3h^{i(1)} h^j \theta_{ij} + h^i h^j h^k \theta_{ijk}\end{aligned}$$

The equality of mixed partial derivatives in these calculations makes it evident that one can use a formalism that hides all the superscripts and subscripts (i, j, k, \dots). In that simplified formalism, we can write

$$\begin{aligned}\theta^{(1)} &= h\theta \\ \theta^{(2)} &= h^{(1)}\theta + h^2\theta \\ \theta^{(3)} &= h^{(2)}\theta + 3h^{(1)}h\theta + h^3\theta \\ \theta^{(4)} &= h^4\theta + 6h^2\theta h^{(1)} + 3\theta h^{(1)2} + 4h\theta h^{(2)} + \theta h^{(3)}\end{aligned}$$

Each successive row is obtained from the previous row by applying the identity $\theta^{(1)} = h\theta$ in conjunction with the product rule for the derivative.

This procedure can be automated so that one can obtain the formulas for higher order derivatives as far as one desires. These can then be converted into the non-commutative constraint algebra and the consequences examined. Further analysis of this kind will be done in a sequel to this paper.

The interested reader may enjoy seeing how this formalism can be carried out. Below we illustrate a calculation using *Mathematica*TM, where the program already knows how to formally differentiate using the product rule and so only needs to be told that $\theta^{(1)} = h\theta$. This is said in the equation $T'[x] = H[x]T[x]$ where $T[x]$ stands for θ and $H[x]$ stands for h with x a dummy variable for the differentiation. Here $D[T[x], x]$ denotes the derivative of $T[x]$ with respect to x , as does $T'[x]$,

In the calculation below we have indicated five levels of derivative. The structure of the coefficients in this recursion is interesting and complex territory. For example, the coefficients of $H[x]^n T[x] H'[x] = h^n \theta h'$ are the triangular numbers $\{1, 3, 6, 10, 15, 21, \dots\}$ but the next series are the coefficients of $H[x]^n T[x] H'[x]^2 = h^n \theta h'^2$, and these form the series

$$\{1, 3, 15, 45, 105, 210, 378, 630, 990, 1485, 2145, \dots\}.$$

This series is eventually constant after four discrete differentiations. This is the next simplest series that occurs in this structure after the triangular numbers. To penetrate the full algebra of constraints we need to understand the structure of these derivatives and their corresponding non-commutative symmetrizations.

$$T'[x] := H[x]T[x]$$

$$\begin{aligned} & D[T[x], x] \\ & D[D[T[x], x], x] \\ & D[D[D[T[x], x], x], x] \\ & D[D[D[D[T[x], x], x], x], x] \\ & D[D[D[D[D[T[x], x], x], x], x], x] \\ & H[x]T[x] \end{aligned}$$

$$H[x]^2T[x] + T[x]H'[x]$$

$$H[x]^3T[x] + 3H[x]T[x]H'[x] + T[x]H''[x]$$

$$H[x]^4T[x] + 6H[x]^2T[x]H'[x] + 3T[x]H'[x]^2 + 4H[x]T[x]H''[x] + T[x]H^{(3)}[x]$$

$$\begin{aligned} & H[x]^5T[x] + 10H[x]^3T[x]H'[x] + 15H[x]T[x]H'[x]^2 + 10H[x]^2T[x]H''[x] + 10T[x]H'[x]H''[x] + \\ & 5H[x]T[x]H^{(3)}[x] + T[x]H^{(4)}[x] \end{aligned}$$

7.1 Algebra of Constraints

In this section we work with the hidden index conventions described before in the paper. In this form, the classical versions of the first two constraint equations are

1. $\dot{\theta} = \theta h$
2. $\ddot{\theta} = \theta h^2 + \theta \dot{h}$

In order to obtain the non-commutative versions of these equations, we replace h by H and θ by Θ where the capitalized versions are non-commuting operators. The first and second constraints then become

1. $\{\dot{\Theta}\} = \{\Theta H\} = \frac{1}{2}(\Theta H + H\Theta)$
2. $\{\ddot{\Theta}\} = \{\Theta H^2\} + \{\Theta \dot{H}\} = \frac{1}{3}(\Theta H^2 + H\Theta H + H^2\Theta) + \frac{1}{2}(\Theta \dot{H} + \dot{H}\Theta)$

Proposition. The Second Constraint is equivalent to the commutator equation

$$[[\Theta, H], H] = 0.$$

Proof. We identify

$$\{\dot{\Theta}\}^\bullet = \{\ddot{\Theta}\}$$

and

$$\{\dot{\Theta}\}^\bullet = \{\{\Theta H\}H\} + \{\Theta \dot{H}\}.$$

So we need

$$\{\Theta H^2\} = \{\{\Theta H\}H\}.$$

The explicit formula for $\{\{\Theta H\}H\}$ is

$$\{\{\Theta H\}H\} = \frac{1}{2}(\{\Theta H\}H + H\{\Theta H\}) = \frac{1}{4}(\theta HH + H\Theta H + H\Theta H + HH\Theta).$$

Thus we require that

$$\frac{1}{3}(\Theta H^2 + H\Theta H + H^2\Theta) = \frac{1}{4}(\theta HH + H\Theta H + H\Theta H + HH\Theta).$$

which is equivalent to

$$\Theta H^2 + H^2\Theta - 2H\Theta H = 0.$$

We then note that

$$[[\Theta, H], H] = (\Theta H - H\Theta)H - H(\Theta H - H\Theta) = \Theta H^2 + H^2\Theta - 2H\Theta H.$$

Thus the final form of the second constraint is the equation

$$[[\Theta, H], H] = 0. //$$

The Third Constraint. We now go on to an analysis of the third constraint. The third constraint consists in the the two equations

1. $\{\ddot{\Theta}\} = \{\Theta H^3\} + 3\{\Theta H\dot{H}\} + \{\Theta\ddot{H}\}$
2. $\{\ddot{\Theta}\} = \{\ddot{\Theta}\}^\bullet$ where

$$\{\ddot{\Theta}\}^\bullet = \{\{\Theta H\}H^2\} + 2\{\Theta H\dot{H}\} + \{\{\Theta H\}\dot{H}\} + \{\Theta\ddot{H}\}$$

Proposition. The Third Constraint is equivalent to the commutator equation

$$[H^2, [H, \Theta]] = [\dot{H}, [H, \Theta]] - 2[H, [\dot{H}, \Theta]].$$

Proof. We demand that $\{\ddot{\Theta}\} = \{\ddot{\Theta}\}^\bullet$ and this becomes the longer equation

$$\{\Theta H^3\} + 3\{\Theta H\dot{H}\} + \{\Theta\ddot{H}\} = \{\{\Theta H\}H^2\} + 2\{\Theta H\dot{H}\} + \{\{\Theta H\}\dot{H}\} + \{\Theta\ddot{H}\}$$

This is equivalent to the equation

$$\{\Theta H^3\} + \{\Theta H\dot{H}\} = \{\{\Theta H\}H^2\} + \{\{\Theta H\}\dot{H}\}$$

This, in turn is equivalent to

$$\{\Theta H^3\} - \{\{\Theta H\}H^2\} = \{\{\Theta H\}\dot{H}\} - \{\Theta H\dot{H}\}$$

This is equivalent to

$$(1/4)(H^3\Theta + H^2\Theta H + H\Theta H^2 + \Theta H^3) - (1/6)(H^2(H\Theta + \Theta H) + H(H\Theta + \Theta H)H + (H\Theta + \Theta H)H^2) \\ = (1/2)(\dot{H}(1/2)(H\Theta + \Theta H) + (1/2)(H\Theta + \Theta H)\dot{H}) - (1/6)(\dot{H}H\Theta + \dot{H}\Theta H + H\dot{H}\Theta + H\Theta\dot{H} + \Theta H\dot{H} + \Theta\dot{H}H)$$

This is equivalent to

$$3(H^3\Theta + H^2\Theta H + H\Theta H^2 + \Theta H^3) - 2(H^3\Theta + 2H^2\Theta H + 2H\Theta H^2 + \Theta H^3) \\ = 3(\dot{H}H\Theta + \dot{H}\Theta H + H\Theta\dot{H} + \Theta H\dot{H}) - 2(\dot{H}H\Theta + \dot{H}\Theta H + H\dot{H}\Theta + H\Theta\dot{H} + \Theta H\dot{H} + \Theta\dot{H}H)$$

This is equivalent to

$$H^3\Theta - H^2\Theta H - H\Theta H^2 + \Theta H^3 \\ = (\dot{H}H\Theta + \dot{H}\Theta H + H\Theta\dot{H} + \Theta H\dot{H}) - 2(H\dot{H}\Theta + \Theta\dot{H}H)$$

The reader can now easily verify that

$$[H^2, [H, \Theta]] = H^3\Theta - H^2\Theta H - H\Theta H^2 + \Theta H^3$$

and that

$$[\dot{H}, [H, \Theta]] - 2[H, [\dot{H}, \Theta]] = (\dot{H}H\Theta + \dot{H}\Theta H + H\Theta\dot{H} + \Theta H\dot{H}) - 2(H\dot{H}\Theta + \Theta\dot{H}H)$$

Thus we have proved that the third constraint equations are equivalent to the commutator equation

$$[H^2, [H, \Theta]] = [\dot{H}, [H, \Theta]] - 2[H, [\dot{H}, \Theta]]$$

This completes the proof of the Proposition. //

Discussion. Each successive constraint involves the explicit formula for the higher derivatives of Θ coupled with the extra constraint that

$$\{\Theta^{(n)}\}^\bullet = \{\Theta^{(n+1)}\}.$$

We conjecture that each constraint can be expressed as a commutator equation in terms of Θ , H and the derivatives of H , in analogy to the formulas that we have found for the first three constraints. This project will continue with a deeper algebraic study of the constraints and their physical meanings.

8 Appendix – Einstein's Equations and the Bianchi Identity

The purpose of this section is to show how the Bianchi identity (see below for its definition) appears in the context of non-commutative worlds. The Bianchi identity is a crucial mathematical ingredient in general relativity. We shall begin with a quick review of the mathematical structure of general relativity (see for example [6]) and then turn to the context of non-commutative worlds.

The basic tensor in Einstein's theory of general relativity is

$$G^{ab} = R^{ab} - \frac{1}{2}Rg^{ab}$$

where R^{ab} is the Ricci tensor and R the scalar curvature. The Ricci tensor and the scalar curvature are both obtained by contraction from the Riemann curvature tensor R_{bcd}^a with $R_{ab} = R_{abc}^c$, $R^{ab} = g^{ai}g^{bj}R_{ij}$, and $R = g^{ij}R_{ij}$. Because the Einstein tensor G^{ab} has vanishing divergence, it is a prime candidate to be proportional to the energy momentum tensor $T^{\mu\nu}$. The Einstein field equations are

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = \kappa T^{\mu\nu}.$$

The reader may wish to recall that the Riemann tensor is obtained from the commutator of a covariant derivative ∇_k , associated with the Levi-Civita connection $\Gamma_{jk}^i = (\Gamma_k)_j^i$ (built from the space-time metric g_{ij}). One has

$$\lambda_{a;b} = \nabla_b \lambda_a = \partial_b \lambda_a - \Gamma_{ab}^d \lambda_d$$

or

$$\lambda_{;b} = \nabla_b \lambda = \partial_b \lambda - \Gamma_b \lambda$$

for a vector field λ . With

$$R_{ij} = [\nabla_i, \nabla_j] = \partial_j \Gamma_i - \partial_i \Gamma_j + [\Gamma_i, \Gamma_j],$$

one has

$$R_{bcd}^a = (R_{cd})_b^a.$$

(Here R_{cd} is *not* the Ricci tensor. It is the Riemann tensor with two internal indices hidden from sight.)

One way to understand the mathematical source of the Einstein tensor, and the vanishing of its divergence, is to see it as a contraction of the Bianchi identity for the Riemann tensor. The Bianchi identity states

$$R_{bcd:e}^a + R_{bde:c}^a + R_{bec:d}^a = 0$$

where the index after the colon indicates the covariant derivative. Note also that this can be written in the form

$$(R_{cd:e})_b^a + (R_{de:c})_b^a + (R_{ec:d})_b^a = 0.$$

The Bianchi identity is a consequence of local properties of the Levi-Civita connection and consequent symmetries of the Riemann tensor. One relevant symmetry of the Riemann tensor is the equation $R_{bcd}^a = -R_{bdc}^a$.

We will not give a classical derivation of the Bianchi identity here, but it is instructive to see how its contraction leads to the Einstein tensor. To this end, note that we can contract the Bianchi identity to

$$R_{bca:e}^a + R_{bae:c}^a + R_{bec:a}^a = 0$$

which, in the light of the above definition of the Ricci tensor and the symmetries of the Riemann tensor is the same as

$$R_{bc:e} - R_{be:c} + R_{bec:a}^a = 0.$$

Contract this tensor equation once more to obtain

$$R_{bc:b} - R_{bb:c} + R_{bbc:a}^a = 0,$$

and raise indices

$$R_{c:b}^b - R_{:c} + R_{bc:a}^{ab} = 0.$$

Further symmetry gives

$$R_{bc:a}^{ab} = R_{cb:a}^{ba} = R_{c:a}^a = R_{c:b}^b.$$

Hence we have

$$2R_{c:b}^b - R_{:c} = 0,$$

which is equivalent to the equation

$$(R_c^b - \frac{1}{2}R\delta_c^b)_{:b} = G_{c:b}^b = 0.$$

From this we conclude that $G_{:b}^{bc} = 0$. The Einstein tensor has appeared on the stage with vanishing divergence, courtesy of the Bianchi identity!

Bianchi Identity and Jacobi Identity. Now lets turn to the context of non-commutative worlds. We have infinitely many possible covariant derivatives, all of the form

$$F_{:a} = \nabla_a F = [F, N_a]$$

for some N_a elements in the non-commutative world. Choose any such covariant derivative. Then, as in the introduction to this paper, we have the curvature

$$R_{ij} = [N_i, N_j]$$

that represents the commutator of the covariant derivative with itself in the sense that $[\nabla_i, \nabla_j]F = [[N_i, N_j], F]$. Note that R_{ij} is not a Ricci tensor, but rather the indication of the external structure of the curvature without any particular choice of linear representation (as is given in the classical case as described above). We then have the Jacobi identity

$$[[N_a, N_b], N_c] + [[N_c, N_a], N_b] + [[N_b, N_c], N_a] = 0.$$

Writing the Jacobi identity in terms of curvature and covariant differentiation we have

$$R_{ab:c} + R_{ca:b} + R_{bc:a} = 0.$$

Thus in a non-commutative world, every covariant derivative satisfies its own Bianchi identity. This gives an impetus to study general relativity in non-commutative worlds by looking for covariant derivatives that satisfy the symmetries of the Riemann tensor and link with a metric in an appropriate way. We have only begun this aspect of the investigation. The point of this section has been to show the intimate relationship between the Bianchi identity and the Jacobi identity that is revealed in the context of non-commutative worlds.

9 Philosophical Appendix

The purpose of this appendix is to point to a way of thinking about the relationship of mathematics, physics, persons, and observations that underlies the approach taken in this paper. We began constructions motivating non-commutativity by considering sequences of actions $\cdots DCBA$ written from right to left so that they could be applied to an actant X in the order $\cdots DCBAX = \cdots (D(C(B(AX)))) \cdots$. The sequence of events A, B, C, D, \cdots was conceptualized as a temporal order, with the events themselves happening at levels or frames of successive “space”. *There is no ambient coordinate space, nor is there any continuum of time.* All that is given is the possibility of structure at any given moment, and the possibility of distinguishing structures from one moment to the next. In this light the formula $DX = [X, J] = XJ - JX = J(X' - X)$ connotes a symbolic representation of the measurement of a difference across one time interval, nothing more. In other words DX represents a difference taken across a background difference (the time step). Once the Pandora’s box of measuring such differences has been opened, we are subject to the multiplicities of forms of difference $\nabla_K X = [X, K]$, their non-commutativity among themselves, the notion of a flat background that has the formal appearance of quantum mechanics, the emergence of abstract curvature and formal gauge fields. All this occurs in these calculi of differences *prior* to the emergence of differential geometry or topology or even the notion of linear superposition of states (so important to quantum mechanics). Note that in this algebraic patterning each algebra element X is an actant (can be acted upon) and an actor (via the operator ∇_X). In Lie algebras, this is the relationship between the algebra and its adjoint representation that makes each element of the algebra into a representor for that algebra by exactly the formula $adj_A(X) = [A, X] = -\nabla_A(X)$ that we have identified as a formal difference or derivative, a generator for a calculus of differences.

The precursor and conceptual background of our particular formalism is therefore the concept of discrimination, the idea of a distinction. A key work in relation to that concept is the book “Laws of Form” by G. Spencer-Brown [28] in which is set out a calculus of distinction of maximal simplicity and generality. In that calculus a mark (denoted here by a bracket $< >$) represents a distinction and is seen to be a distinction between inside and outside. In this elemental mathematics there is no distinction except the one that we draw between the mathematician and the operator in the formal system as sign/symbol/interpretant. This gives full responsibility to the mathematician to draw the boundaries between the formal system as physical interaction and the formal system as symbolic entity and the formal system as Platonic conceptual form. In making a mathematics of distinction, the mathematician tells a story to himself/herself about the creation of a world. Spencer-Brown’s iconic mathematics can be extended to contact any mathematics, and when this happens that mathematics is transformed into a personal creation of the mathematician who uses it. In a similar (but to a mathematician) darker way, the physicist is intimately bound to the physical reality that he studies.

We could have begun this paper with the the Spencer-Brown mark as bracket: $< >$. This empty bracket is seen to make a distinction between inside and outside. In order for that to occur the bracket has to become a process in the perception of someone. It has to leave whatever

objective existence or potentiality it has alone (all one) and become the locus or nexus of an idea in a perceiving mind. As such it is stabilized by that perception/creation and becomes really a solution to $\{< >\} = < >$ where the curly bracket (the form of perception) is in the first place identical to the mark $< >$, and then distinguished from it by the act of distinguishing world and perceiver. It is within this cleft of the infinite recursive and the finite

$$< > = \{< >\} = \{\{< >\}\} = \{\{\{< >\}\}\} = \dots = \{\{\{\{\{\dots\}\}\}\}\}$$

that the objectivity of mathematics/physics (they are not different in the cleft) arises. All the rest of mathematics or calculus of brackets needs come forth for the observer in the same way. Through that interaction there is the possibility of a deep dialogue of many levels, a dialogue where it is seen that mathematics and physics develop in parallel, each describing the same boundary from opposite sides. That boundary is the imaginary boundary between the inner and outer worlds.

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